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# Fractional Hermite-Hadmard inequalities for convex functions and applications

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**Abstract**

In this paper, we derive a new lemma including third-order derivative of a function via fractional integrals. Using this lemma, we establish some new fractional estimates for Hermite-Hadamard type inequalities for convex functions. Several special cases are also discussed. Some applications to special means of real numbers are also discussed. The ideas and techniques used in this paper may stimulate future investigations regarding Hermite-Hadamard type of inequalities and its application in different areas.

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*Keywords.* Convex functions, fractional integrals, Hermite-Hadamard inequality, means.

## 1 Introduction

A convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ with $a < b$ and $a, b \in I$ always obey the following double inequality which is known as Hermite-Hadamard inequality in the literature

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For useful details on Hermite-Hadamard type of integral inequalities, readers are referred to [2]. For some recent investigations on these inequalities, see [1, 3, 4, 5, 7–18]. In [14] authors proved Hermite-Hadamard type inequalities for fractional integrals. This result inspired many researchers to study and investigate Hermite-Hadmard type of inequalities under the perspective of fractional integrals, for example, see [8, 12, 13, 16].

In this paper, we derive a new result for a three times differentiable function involving fractional integrals. Then, using this result, we obtain several new fractional estimates of Hermite-Hadamard inequalities via convex functions. In last section, we give some applications to special means. We hope that the ideas conveyed in this paper may encourage interested readers to explore new dimensions of research in the field of mathematical inequalities. This is the main motivation of this paper.

## 2 Preliminaries

In this section, we recall some previously known concepts. First of all let set of real numbers be denoted by $\mathbb{R}$. Let $I = [a, b] \subset \mathbb{R}$ be the interval and $I^\circ$ be the interior of $I$. We follow these notations throughout the paper unless otherwise specified.

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**Definition 2.1.** A function $f : I \to \mathbb{R}$ is said to be classical convex function, if

$$f((1-t)x+ty) \leq (1-t)f(x)+tf(y), \quad \forall x,y \in I, t \in [0,1]. \tag{2.1}$$

**Definition 2.2** ([6])**.** Let $f \in L_1[a,b]$. Then Riemann-Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

and

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}x^{\alpha-1}dx,$$

is the Gamma function.

**Definition 2.3** ([2])**.** Recall the following definitions:

1. For arbitrary $a,b \in \mathbb{R} \setminus \{0\}$ and $a \neq b$

$$L(b,a) = \frac{b-a}{\log b - \log a},$$

is the logarithmic mean,

2. For arbitrary $a,b \in \mathbb{R}$ and $a \neq b$

$$A(a,b) = \frac{a+b}{2},$$

is the arithmetic mean,

3. The extended logarithmic mean $L_p$ of two positive numbers $a,b$ is given for $a = b$ by $L_p(a,a) = a$ and for $a \neq b$ by

$$L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0, \\ \dfrac{b-a}{\log b - \log a}, & p = -1, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p = 0. \end{cases}$$

## 3 Main Results

In this section, we establish our main results. To prove our main results, we need following lemma.

**Lemma 3.1.** Let $f : I \to \mathbb{R}$ be three times differentiable function on $I^\circ$. If $f''' \in L[a,b]$, then, following equality for fractional integrals hold:

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{(b-a)^3}{4(\alpha+1)(\alpha+2)}f''\left(\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)$$
$$=\frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left[-f'''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f'''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt.$$

*Proof.* Let

$$\begin{aligned}
I&=\int_0^1(1-t)^{\alpha+2}\left[-f'''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f'''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt\\
&=-\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt+\int_0^1(1-t)^{\alpha}f'''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\\
&=-I_1+I_2. \qquad (3.1)
\end{aligned}$$

Integrating $I_1$ on $[0,1]$, we have

$$\begin{aligned}
I_1&=\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
&=\left|-\frac{2(1-t)^{\alpha+2}f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)}{b-a}\right|_0^1-\frac{2(\alpha+2)}{b-a}\int_0^1(1-t)^{\alpha+1}f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
&=\frac{2}{b-a}f''\left(\frac{a+b}{2}\right)-\frac{2(\alpha+2)}{b-a}\left[\frac{2}{b-a}f'\left(\frac{a+b}{2}\right)\right.\\
&\qquad\left.-\frac{2(\alpha+1)}{b-a}\int_0^1(1-t)^{\alpha}f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right]\\
&=\frac{2}{b-a}f''\left(\frac{a+b}{2}\right)-\frac{4(\alpha+2)}{(b-a)^2}f'\left(\frac{a+b}{2}\right)\\
&\qquad+\frac{4(\alpha+1)(\alpha+2)}{(b-a)^2}\int_0^1(1-t)^{\alpha}f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
&=\frac{2}{b-a}f''\left(\frac{a+b}{2}\right)-\frac{4(\alpha+2)}{(b-a)^2}f'\left(\frac{a+b}{2}\right)\\
&\quad+\frac{4(\alpha+1)(\alpha+2)}{(b-a)^2}\left[\frac{2}{b-a}f\left(\frac{a+b}{2}\right)-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{b-a} f''\left(\frac{a+b}{2}\right) - \frac{4(\alpha+2)}{(b-a)^2} f'\left(\frac{a+b}{2}\right) \\
&\quad + \frac{8(\alpha+1)(\alpha+2)}{(b-a)^3} f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+3}((\alpha+1)(\alpha+2))\Gamma(\alpha+1)}{(b-a)^{\alpha+3}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) \\
&= \frac{2}{b-a} f''\left(\frac{a+b}{2}\right) - \frac{4(\alpha+2)}{(b-a)^2} f'\left(\frac{a+b}{2}\right) \\
&\quad + \frac{8(\alpha+1)(\alpha+2)}{(b-a)^3} f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+3}\Gamma(\alpha+3)}{(b-a)^{\alpha+3}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a).
\end{aligned} \tag{3.2}
$$

In a similar way, integrating $I_2$ on $[0,1]$, we have

$$
\begin{aligned}
I_2 &= \int_0^1 (1-t)^{\alpha+2} f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \\
&= \left| \frac{2(1-t)^{\alpha+2} f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)}{b-a} \right|_0^1 + \frac{2(\alpha+2)}{b-a} \int_0^1 (1-t)^{\alpha+1} f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \\
&= -\frac{2}{b-a} f''\left(\frac{a+b}{2}\right) + \frac{2(\alpha+2)}{b-a} \left[ -\frac{2}{b-a} f'\left(\frac{a+b}{2}\right) \right. \\
&\qquad \left. + \frac{2(\alpha+1)}{b-a} \int_0^1 (1-t)^{\alpha} f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \right] \\
&= -\frac{2}{b-a} f''\left(\frac{a+b}{2}\right) - \frac{4(\alpha+2)}{(b-a)^2} f'\left(\frac{a+b}{2}\right) \\
&\qquad + \frac{4(\alpha+1)(\alpha+2)}{(b-a)^2} \int_0^1 (1-t)^{\alpha} f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \\
&= -\frac{2}{b-a} f''\left(\frac{a+b}{2}\right) - \frac{4(\alpha+2)}{(b-a)^2} f'\left(\frac{a+b}{2}\right) \\
&\quad + \frac{4(\alpha+1)(\alpha+2)}{(b-a)^2} \left[ -\frac{2}{b-a} f\left(\frac{a+b}{2}\right) + \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b) \right] \\
&= -\frac{2}{b-a} f''\left(\frac{a+b}{2}\right) - \frac{4(\alpha+2)}{(b-a)^2} f'\left(\frac{a+b}{2}\right) \\
&\quad - \frac{8(\alpha+1)(\alpha+2)}{(b-a)^3} f\left(\frac{a+b}{2}\right) + \frac{2^{\alpha+3}((\alpha+1)(\alpha+2))\Gamma(\alpha+1)}{(b-a)^{\alpha+3}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b) \\
&= -\frac{2}{b-a} f''\left(\frac{a+b}{2}\right) - \frac{4(\alpha+2)}{(b-a)^2} f'\left(\frac{a+b}{2}\right) \\
&\quad - \frac{8(\alpha+1)(\alpha+2)}{(b-a)^3} f\left(\frac{a+b}{2}\right) + \frac{2^{\alpha+3}\Gamma(\alpha+3)}{(b-a)^{\alpha+3}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)
\end{aligned} \tag{3.3}
$$

Summation of (3.2), (3.3) and (3.1) and then multiplying both sides by $\frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}$ completes the proof. Q.E.D.

Note that for $\alpha = 1$ in Lemma 3.1, we have previously known result [18].

**Theorem 3.2.** Let $f : I \to \mathbb{R}$ be three times differentiable function on $I^\circ$. If $f''' \in L[a,b]$ and $|f'''|$ is convex function, then

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right] - \frac{(b-a)^3}{4(\alpha+1)(\alpha+2)}f''\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^3}{8(\alpha+1)(\alpha+2)(\alpha+3)}\left[|f'''(a)| + |f'''(b)|\right].$$

*Proof.* Using Lemma 3.1 and the fact that $|f'''|$ is convex function, we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right] - \frac{(b-a)^3}{4(\alpha+1)(\alpha+2)}f''\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right)\right|$$
$$= \left|\frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2}\left[-f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right]dt\right|$$
$$\leq \left|\frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2} f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt\right|$$
$$+ \left|\frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2} f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt\right|$$
$$\leq \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2}\left|f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|dt$$
$$+ \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2}\left|f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right|dt$$
$$\leq \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2}\left[\left(\frac{1+t}{2}\right)|f'''(a)| + \left(\frac{1-t}{2}\right)|f'''(b)|\right]dt$$
$$+ \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2}\left[\left(\frac{1-t}{2}\right)|f'''(a)| + \left(\frac{1+t}{2}\right)|f'''(b)|\right]dt$$

$$= \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\left[\left(\frac{\alpha+5}{(\alpha+3)(\alpha+4)}\right)|f'''(a)| + \left(\frac{1}{\alpha+4}\right)|f'''(b)|\right]$$
$$+ \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\left[\left(\frac{1}{\alpha+4}\right)|f'''(a)| + \left(\frac{\alpha+5}{(\alpha+3)(\alpha+4)}\right)|f'''(b)|\right]$$
$$= \frac{(b-a)^3}{8(\alpha+1)(\alpha+2)(\alpha+3)}\left[|f'''(a)| + |f'''(b)|\right].$$

This completes the proof. Q.E.D.

**Theorem 3.3.** Let $f : I \to \mathbb{R}$ be three times differentiable function on $I^\circ$. If $f''' \in L[a,b]$ and $|f'''|^q$ is convex function where $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \geq 1$, then

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b)\right] - \frac{(b-a)^3}{4(\alpha+1)(\alpha+2)} f''\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{3}{4}|f'''(a)|^q + \frac{1}{2}|f'''(b)|^q\right)^{\frac{1}{q}} + \left(\frac{1}{2}|f'''(a)|^q + \frac{3}{4}|f'''(b)|^q\right)^{\frac{1}{q}}\right].$$

*Proof.* Using Lemma 3.1, well known Holder's inequality and the fact that $|f'''|^q$ is convex function, we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b)\right] - \frac{(b-a)^3}{4(\alpha+1)(\alpha+2)} f''\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right)\right|$$
$$= \left|\frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2}\left[-f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right]dt\right|$$
$$\leq \left|\frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2} f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt\right|$$
$$+ \left|\frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2} f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt\right|$$
$$\leq \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2}\left|f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|dt$$
$$+ \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2}\left|f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right|dt$$

$$\leq \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)} \left( \int_0^1 (1-t)^{p(\alpha+2)} dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f''' \left( \frac{1+t}{2} a + \frac{1-t}{2} b \right) \right|^q dt \right)^{\frac{1}{q}}$$
$$+ \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)} \left( \int_0^1 (1-t)^{p(\alpha+2)} dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f''' \left( \frac{1-t}{2} a + \frac{1+t}{2} b \right) \right|^q dt \right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)} \left( \frac{1}{p(\alpha+2)+1} \right)^{\frac{1}{p}} \left( \frac{3}{4} |f'''(a)|^q + \frac{1}{2} |f'''(b)|^q \right)^{\frac{1}{q}}$$
$$+ \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)} \left( \frac{1}{p(\alpha+2)+1} \right)^{\frac{1}{p}} \left( \frac{1}{2} |f'''(a)|^q + \frac{3}{4} |f'''(b)|^q \right)^{\frac{1}{q}}$$
$$= \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)} \left( \frac{1}{p(\alpha+2)+1} \right)^{\frac{1}{p}}$$
$$\times \left[ \left( \frac{3}{4} |f'''(a)|^q + \frac{1}{2} |f'''(b)|^q \right)^{\frac{1}{q}} + \left( \frac{1}{2} |f'''(a)|^q + \frac{3}{4} |f'''(b)|^q \right)^{\frac{1}{q}} \right].$$

This completes the proof. Q.E.D.

**Theorem 3.4.** Let $f : I \to \mathbb{R}$ be three times differentiable function on $I^\circ$. If $f''' \in L[a,b]$ and $|f'''|^q$ is convex function where $q > 1$, then

$$\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) \right] - \frac{(b-a)^3}{4(\alpha+1)(\alpha+2)} f'' \left( \frac{a+b}{2} \right) - f \left( \frac{a+b}{2} \right) \right|$$
$$\leq \frac{(b-a)^3(\alpha+3)}{2^{q+4}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}$$
$$\times \left[ \left( \frac{\alpha+5}{\alpha+3} |f'''(a)|^q + |f'''(b)|^q \right)^{\frac{1}{q}} + \left( |f'''(a)|^q + \frac{\alpha+5}{\alpha+3} |f'''(b)|^q \right)^{\frac{1}{q}} \right].$$

*Proof.* Using Lemma 3.1, well known power mean inequality and the fact that $|f'''|^q$ is convex function, we have

$$\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) \right] - \frac{(b-a)^3}{4(\alpha+1)(\alpha+2)} f'' \left( \frac{a+b}{2} \right) - f \left( \frac{a+b}{2} \right) \right|$$
$$= \left| \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)} \int_0^1 (1-t)^{\alpha+2} \left[ -f''' \left( \frac{1+t}{2} a + \frac{1-t}{2} b \right) + f''' \left( \frac{1-t}{2} a + \frac{1+t}{2} b \right) \right] dt \right|$$

$$
\begin{aligned}
&\leq \left|\frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2} f'''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right| \\
&\qquad + \left|\frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 (1-t)^{\alpha+2} f'''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right| \\
&\leq \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\left(\int_0^1 (1-t)^{(\alpha+2)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^{(\alpha+2)}\left|f'''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\left(\int_0^1 (1-t)^{(\alpha+2)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^{(\alpha+2)}\left|f'''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^3}{2^{q+4}(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}\left(\frac{\alpha+5}{(\alpha+3)(\alpha+4)}|f'''(a)|^q+\frac{1}{\alpha+4}|f'''(b)|^q\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-a)^3}{2^{q+4}(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+4}|f'''(a)|^q+\frac{\alpha+5}{(\alpha+3)(\alpha+4)}|f'''(b)|^q\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^3(\alpha+3)}{2^{q+4}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} \\
&\quad \times \left[\left(\frac{\alpha+5}{\alpha+3}|f'''(a)|^q+|f'''(b)|^q\right)^{\frac{1}{q}}+\left(|f'''(a)|^q+\frac{\alpha+5}{\alpha+3}|f'''(b)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

This completes the proof. Q.E.D.

## 4 Some Applications to special means

In this section, we present some applications to means of real numbers.

**Proposition 4.1.** For some $n \in \mathbb{Z}\ \{-1, 0\}$, $0 \leq a < b$, then

$$
\begin{aligned}
&\left|L(a,b)-\frac{n(n-1)(b-a)^3}{24}A^{n-2}(a,b)-A^n(a,b)\right| \\
&\leq \frac{n(n-1)(n-2)(b-a)^3}{192}\left[|a|^{n-3}+|b|^{n-3}\right].
\end{aligned}
$$

*Proof.* The assertion directly follows from Theorem 3.2 applying for $f(x) = x^n$ and $\alpha = 1$. Q.E.D.

**Proposition 4.2.** For some $n \in \mathbb{Z}\ \{-1,0\}$, $0 \le a < b$ and $\frac{1}{p}+\frac{1}{q}=1$, $1<q<\infty$, then

$$\left|L(a,b) - \frac{n(n-1)(b-a)^3}{24}A^{n-2}(a,b) - A^n(a,b)\right|$$
$$\le \frac{n(n-1)(n-2)(b-a)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{3}{4}|a|^{q(n-3)}+\frac{1}{2}|b|^{q(n-3)}\right)^{\frac{1}{q}}+\left(\frac{1}{2}|a|^{q(n-3)}+\frac{3}{4}|b|^{q(n-3)}\right)^{\frac{1}{q}}\right].$$

*Proof.* The assertion directly follows from Theorem 3.3 applying for $f(x)=x^n$ and $\alpha=1$. Q.E.D.

**Proposition 4.3.** For some $n \in \mathbb{Z}\ \{-1,0\}$, $0 \le a < b$ and $q>1$, then

$$\left|L(a,b) - \frac{n(n-1)(b-a)^3}{24}A^{n-2}(a,b) - A^n(a,b)\right|$$
$$\le \frac{n(n-1)(n-2)(b-a)^3}{384}\left(\frac{4}{5}\right)^{\frac{1}{q}}$$
$$\times\left[\left(\frac{3}{2}|a|^{q(n-3)}+|b|^{q(n-3)}\right)^{\frac{1}{q}}+\left(|a|^{q(n-3)}+\frac{3}{2}|b|^{q(n-3)}\right)^{\frac{1}{q}}\right].$$

*Proof.* The assertion directly follows from Theorem 3.4 applying for $f(x)=x^n$ and $\alpha=1$. Q.E.D.

**Proposition 4.4.** For some $0 \le a < b$, then

$$\left|L^{-1}(a,b) - \frac{(b-a)^3}{12}A^{-3}(a,b) - A^{-1}(a,b)\right| \le \frac{(b-a)^3}{32}\left[|a|^{-4}+|b|^{-4}\right].$$

*Proof.* The assertion directly follows from Theorem 3.2 applying for $f(x)=x^{-1}$ and $\alpha=1$. Q.E.D.

**Proposition 4.5.** For some $0 \le a < b$ and $\frac{1}{p}+\frac{1}{q}=1$, $1<q<\infty$, then

$$\left|L^{-1}(a,b) - \frac{(b-a)^3}{12}A^{-3}(a,b) - A^{-1}(a,b)\right|$$
$$\le \frac{(b-a)^3}{16}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3}{4}|a|^{-4q}+\frac{1}{2}|b|^{-4q}\right)^{\frac{1}{q}}+\left(\frac{1}{2}|a|^{-4q}+\frac{3}{4}|b|^{-4q}\right)^{\frac{1}{q}}\right].$$

*Proof.* The assertion directly follows from Theorem 3.3 applying for $f(x)=x^{-1}$ and $\alpha=1$. Q.E.D.

**Proposition 4.6.** For some $0 \le a < b$ and $q>1$, then

$$\left|L^{-1}(a,b) - \frac{(b-a)^3}{12}A^{-3}(a,b) - A^{-1}(a,b)\right|$$
$$\le \frac{(b-a)^3}{64}\left(\frac{4}{5}\right)^{\frac{1}{q}}\left[\left(\frac{3}{2}|a|^{-4q}+|b|^{-4q}\right)^{\frac{1}{q}}+\left(|a|^{-4q}+\frac{3}{2}|b|^{-4q}\right)^{\frac{1}{q}}\right].$$

*Proof.* The assertion directly follows from Theorem 3.4 applying for $f(x) = x^{-1}$ and $\alpha = 1$. Q.E.D.

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